Patriots Over Israel

Approved Solution by M. Heinen

Intermediate Algebra – Computer Lab

(Minimum project requirements)

December 8, 2010

1. **Problem Statement**

Given the parameters specified in the problem statement (see Appendix D for a clean copy of this problem statement), determine:

- Case I – The earliest intercept time of the IRBM by the Patriot
- Case II – The latest possible launch time and intercept time of the IRBM by the Patriot
- Case III – The time of impact assuming the IRBM is NOT intercepted.

Assume t = 0 at IRBM acquisition.

2. **Solution Technique:**

- Use MS Excel and the equation provided in the Appendix A problem statement to calculate the altitudes of the IRBM and Patriot warheads as a function of time t (in seconds). Analyze these altitudes for each of the cases: Case I, Case II, and Case III to determine solutions to each of the cases.
- Use Mathcad 15.0 to analytically solve the Case I, Case II, and Case III equations for increased accuracy.

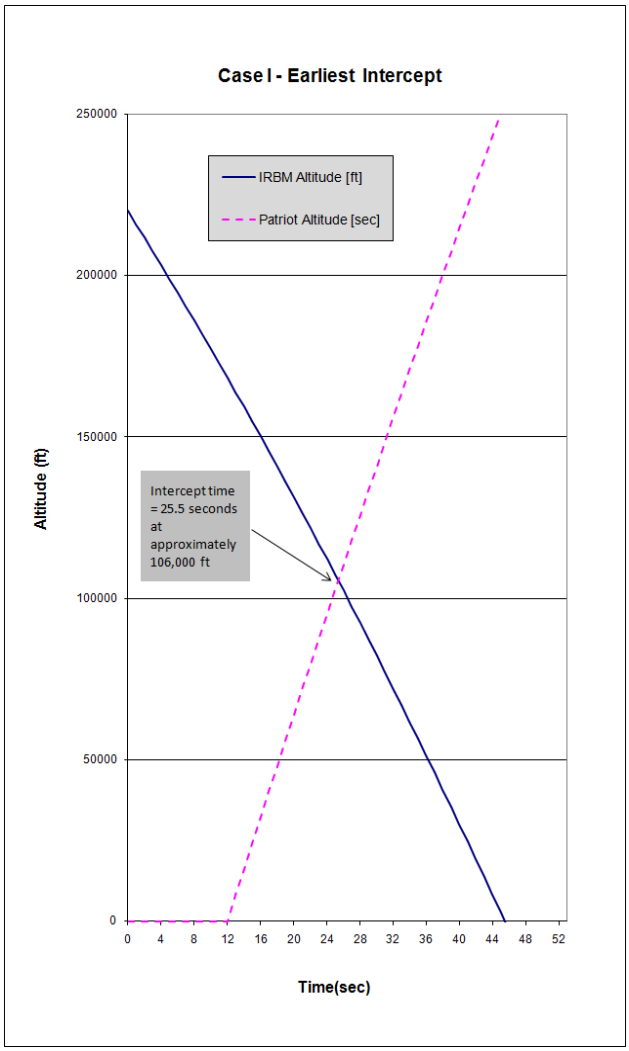
3. **Analysis:**

Case I – Earliest intercept with a Patriot missile launch at t = 12 seconds after target acquisition.

The MS Excel solution graph is shown in Figure 1 and detailed in Appendix A along with Mathcad solutions

Figure 1 shows an early IRMB interception at approximately 106,000 ft at t = 25.5 seconds.

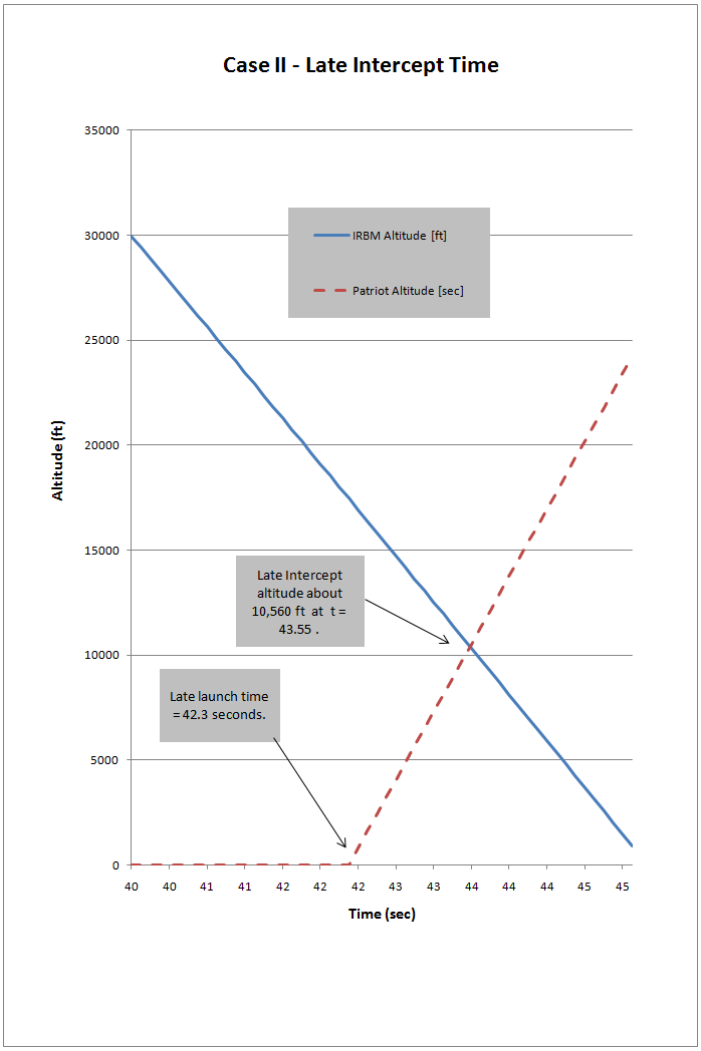
Figure 1



Case II – Latest launch MS Excel graph is show in Figure 2 with Appendix B detailing the Excel and Mathcad solutions.

Late launch time 42.3 seconds with an intercept at approximately 10,600 ft (2 miles).

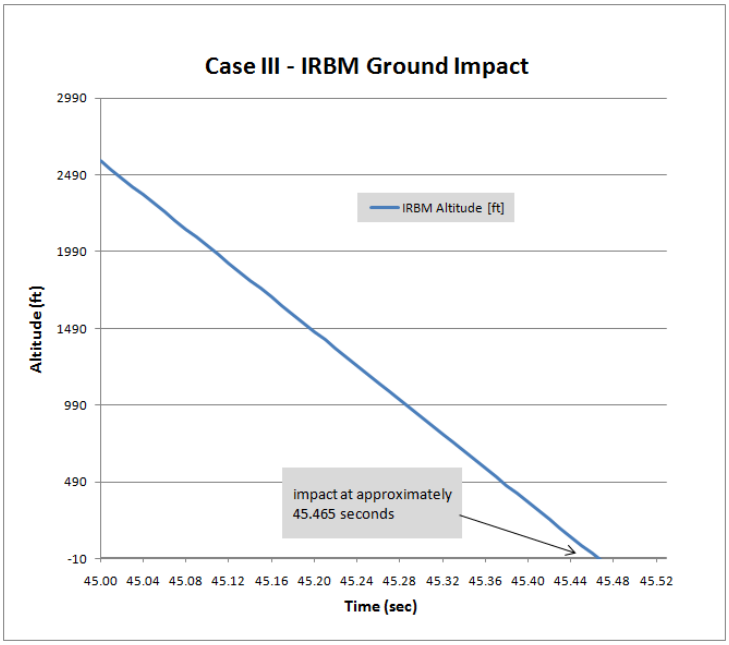
Figure 2



Case III – Ground impact time for undestroyed warhead is shown in Figure 3

Impact time = 45.46 seconds assuming the IRBM warhead is not intercepted.

Figure 3



4. Conclusions / Answers

Table 1 summarizes solutions found numerically by MS Excel and analytically by Mathcad.

Table 1 - Solution Summary

Case	MS Excel (numerical) Solution	Mathcad (analytical) Solution
Case I	Impact time = 25.5 s Impact altitude = 106,000 ft	Impact time = 25.356 s Impact altitude = 105517.851 ft
Case II	Late launch time = 42.3 s	Late launch time = 42.2579 s
Case III	Ground impact time = 45.465 s	Ground impact time = 45.469 s

5. Final Comments

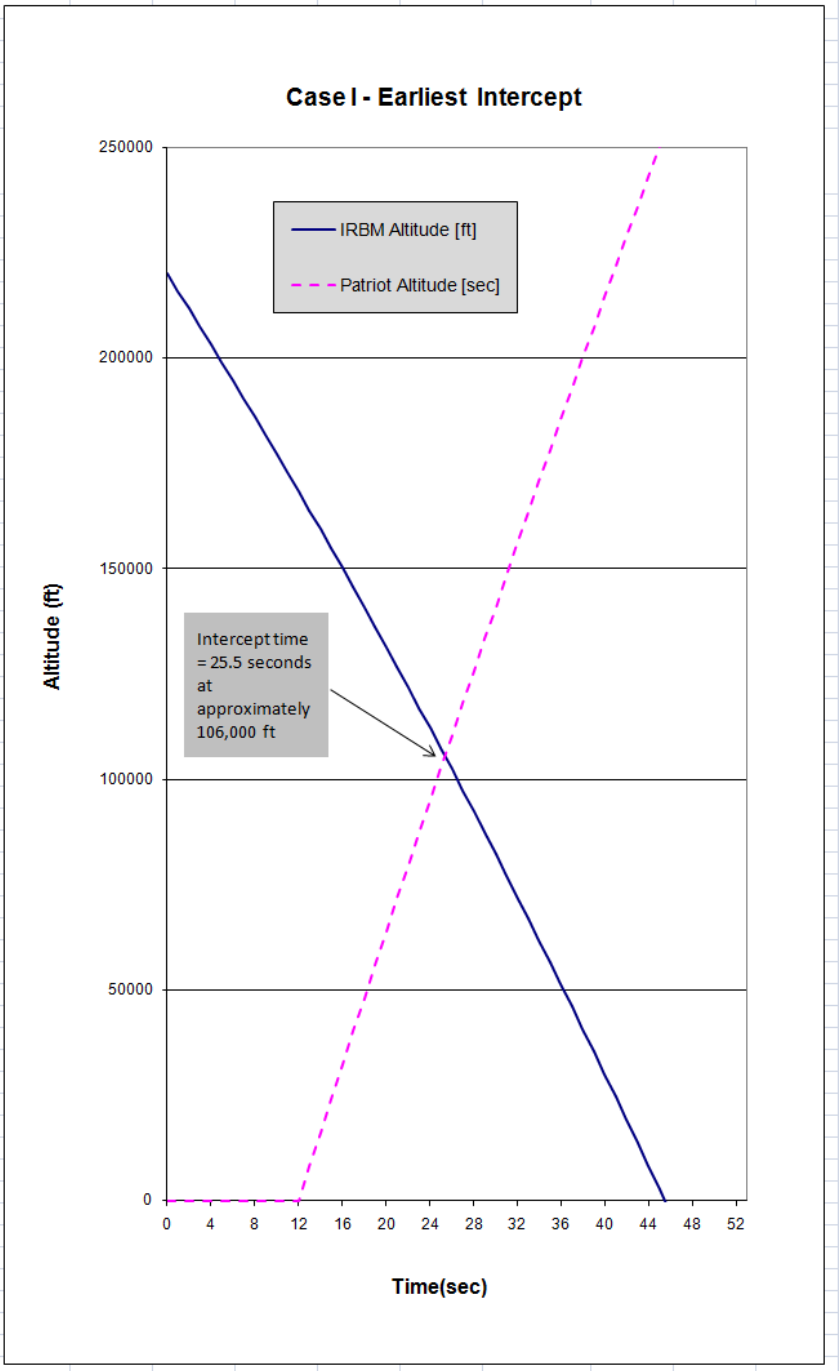
(students – you could say a lot here about your findings)

Appendices

Appendix A – Case I

t start (sec) =	0		units
t inc (sec) =	1		sec
launch delay (sec) =		12	sec
initial height (ft) =	220000	0	ft
initial vel (ft/sec) =	-4107	8115	ft/sec

Time (IRBM) [sec]	IRBM Altitude [ft]	Time (Patriot) [sec]	Patriot Altitude [sec]	diff (root)
0.00	220000	0.00	0	220000
1.00	215877	0.00	0	215877
2.00	211722	0.00	0	211722
3.00	207534	0.00	0	207534
4.00	203314	0.00	0	203314
5.00	199063	0.00	0	199063
6.00	194778	0.00	0	194778
7.00	190462	0.00	0	190462
8.00	186114	0.00	0	186114
9.00	181733	0.00	0	181733
10.00	177320	0.00	0	177320
11.00	172875	0.00	0	172875
12.00	168398	0.00	0	168398
13.00	163888	1.00	8099	155789
14.00	159346	2.00	16166	143181
15.00	154773	3.00	24200	130572
16.00	150166	4.00	32202	117964
17.00	145528	5.00	40173	105356
18.00	140858	6.00	48110	92747
19.00	136155	7.00	56016	80139
20.00	131420	8.00	63890	67530
21.00	126653	9.00	71731	54922
22.00	121854	10.00	79540	42314
23.00	117022	11.00	87317	29705
24.00	112158	12.00	95062	17097
25.00	107263	13.00	102774	4488
26.00	102334	14.00	110454	-8120
27.00	97374	15.00	118103	-20728
28.00	92382	16.00	125718	-33337
29.00	87357	17.00	133302	-45945
30.00	82300	18.00	140854	-58554
31.00	77211	19.00	148373	-71162
32.00	72090	20.00	155860	-83770
33.00	66936	21.00	163315	-96379
34.00	61750	22.00	170738	-108987
35.00	56533	23.00	178128	-121596
36.00	51282	24.00	185486	-134204
37.00	46000	25.00	192813	-146812
38.00	40686	26.00	200106	-159421
39.00	35339	27.00	207368	-172029
40.00	29960	28.00	214598	-184638
41.00	24549	29.00	221795	-197246
42.00	19106	30.00	228960	-209854
43.00	13630	31.00	236093	-222463
44.00	8122	32.00	243194	-235071
45.00	2583	33.00	250262	-247680
46.00	-2990	34.00	257298	-260288
47.00	-8594	35.00	264303	-272896
48.00	-14230	36.00	271274	-285505
49.00	-19899	37.00	278214	-298113
50.00	-25600	38.00	285122	-310722
51.00	-31333	39.00	291997	-323330
52.00	-37098	40.00	298840	-335938
53.00	-42896	41.00	305651	-348547

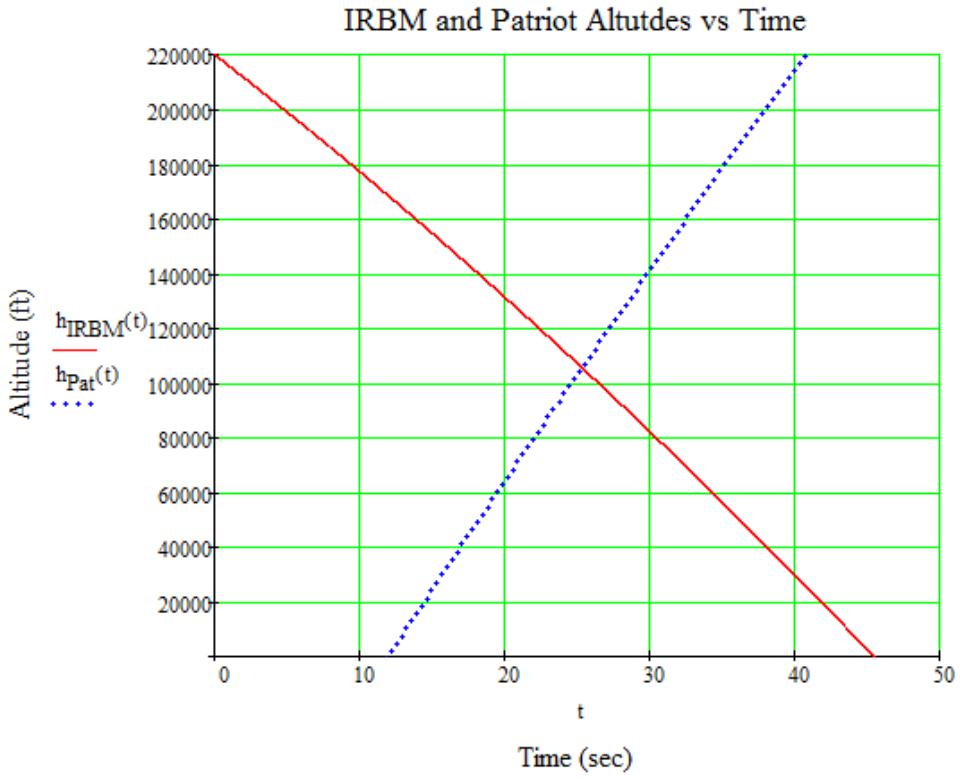


Case I – Mathcad Solution

$v_{oIRBM} := -4107 \cdot \frac{ft}{sec}$ $s_{oIRBM} := 220000 \cdot ft$ $h_{IRBM}(t) := \frac{-g}{2} \cdot t^2 + v_{oIRBM} \cdot t + s_{oIRBM}$

$v_{oPat} := 8115 \cdot \frac{ft}{sec}$ $s_{oPat} := 0 \cdot ft$ $h_{Pat}(t) := \frac{-g}{2} \cdot (t - 12 \cdot s)^2 + v_{oPat} \cdot (t - 12 \cdot s) + s_{oPat}$

$t := 0 \cdot s, .001 \cdot s .. 46 \cdot s$



Case I

Numerical solution $t := 20 \cdot s$ $Case\_I := root\left(h_{IRBM}(t) - h_{Pat}(t), t\right) = 25.356\,s$

$h_{Pat}(Case\_I) = 105517.851\,ft$

Analytical solution: $h_{IRBM}(t) = h_{Pat}(t)$

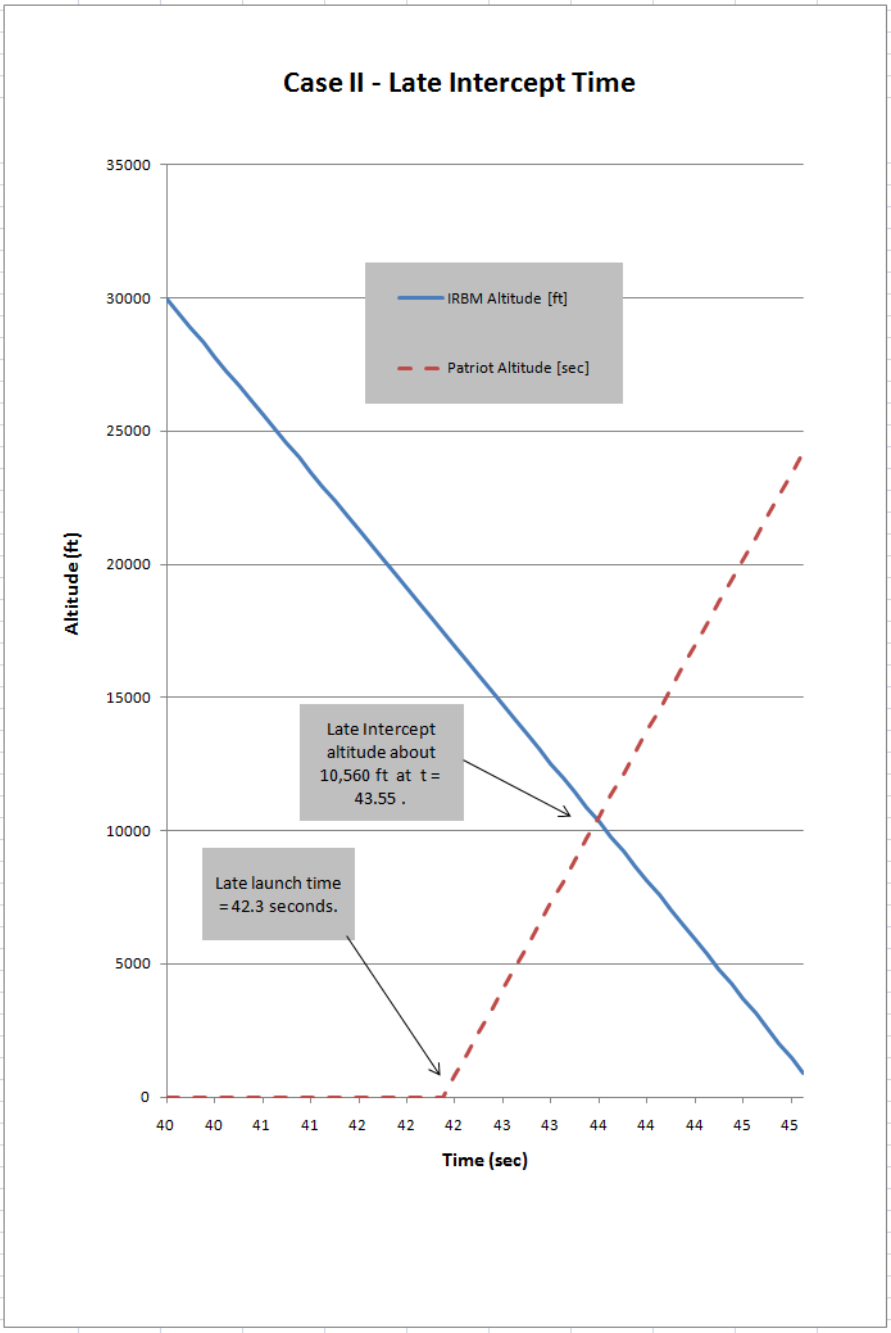
$$\frac{-g}{2} \cdot t^2 + v_{oIRBM} \cdot t + s_{oIRBM} = \frac{-g}{2} \cdot (t - 12 \cdot s)^2 + v_{oPat} \cdot (t - 12 \cdot s) + s_{oPat}$$

solving for t $$t := \frac{72 \cdot g \cdot s^2 + 12 \cdot v_{oPat} \cdot s - s_{oPat} + s_{oIRBM}}{v_{oPat} - v_{oIRBM} + 12 \cdot g \cdot s} = 25.356\,s$$

Appendix B – Case II

t start (sec) =	40		units	10560
t inc (sec) =	0.1		sec	
launch delay (sec) =		42.3	sec	
initial height (ft) =	220000	0	ft	
initial vel (ft/sec) =	-4107	8115	ft/sec	

Time (IRBM) [sec]	IRBM Altitude [ft]	Time (Patriot) [sec]	Patriot Altitude [sec]	diff
40.00	29960	0.00	0	29960
40.10	29420	0.00	0	29420
40.20	28880	0.00	0	28880
40.30	28340	0.00	0	28340
40.40	27799	0.00	0	27799
40.50	27258	0.00	0	27258
40.60	26717	0.00	0	26717
40.70	26176	0.00	0	26176
40.80	25634	0.00	0	25634
40.90	25091	0.00	0	25091
41.00	24549	0.00	0	24549
41.10	24006	0.00	0	24006
41.20	23463	0.00	0	23463
41.30	22919	0.00	0	22919
41.40	22375	0.00	0	22375
41.50	21831	0.00	0	21831
41.60	21287	0.00	0	21287
41.70	20742	0.00	0	20742
41.80	20197	0.00	0	20197
41.90	19651	0.00	0	19651
42.00	19106	0.00	0	19106
42.10	18559	0.00	0	18559
42.20	18013	0.00	0	18013
42.30	17466	0.00	0	17466
42.40	16919	0.10	811	16108
42.50	16372	0.20	1622	14750
42.60	15824	0.30	2433	13391
42.70	15276	0.40	3243	12033
42.80	14728	0.50	4053	10674
42.90	14179	0.60	4863	9316
43.00	13630	0.70	5673	7957
43.10	13081	0.80	6482	6599
43.20	12531	0.90	7290	5241
43.30	11981	1.00	8099	3882
43.40	11431	1.10	8907	2524
43.50	10880	1.20	9715	1165
43.60	10329	1.30	10522	-193
43.70	9778	1.40	11329	-1551
43.80	9227	1.50	12136	-2910
43.90	8675	1.60	12943	-4268
44.00	8122	1.70	13749	-5627
44.10	7570	1.80	14555	-6985
44.20	7017	1.90	15360	-8343
44.30	6464	2.00	16166	-9702
44.40	5910	2.10	16970	-11060
44.50	5356	2.20	17775	-12419
44.60	4802	2.30	18579	-13777
44.70	4248	2.40	19383	-15135
44.80	3693	2.50	20187	-16494
44.90	3138	2.60	20990	-17852
45.00	2582	2.70	21793	-19211
45.10	2027	2.80	22596	-20569
45.20	1471	2.90	23398	-21927
45.30	914	3.00	24200	-23286



Case II Height of IRBM at 2 miles: $2 \cdot mi = \frac{-g}{2} \cdot t^2 + v_{oIRBM} \cdot t + s_{oIRBM}$

$$t_{IRBM2mi} := \begin{bmatrix} \frac{2 \cdot \left(\frac{v_{oIRBM}}{2} + \frac{\sqrt{v_{oIRBM}^2 - 4 \cdot g \cdot mi + 2 \cdot g \cdot s_{oIRBM}}}{2} \right)}{g} \\ -\frac{2 \cdot \left(\frac{\sqrt{v_{oIRBM}^2 - 4 \cdot g \cdot mi + 2 \cdot g \cdot s_{oIRBM}}}{2} - \frac{v_{oIRBM}}{2} \right)}{g} \end{bmatrix} = \begin{pmatrix} 43.56261 \\ -298.86153 \end{pmatrix} s$$

using the principle root: $t_{ILate} := 43.56261 \cdot s$ **where d is the DELAY from t = 0**

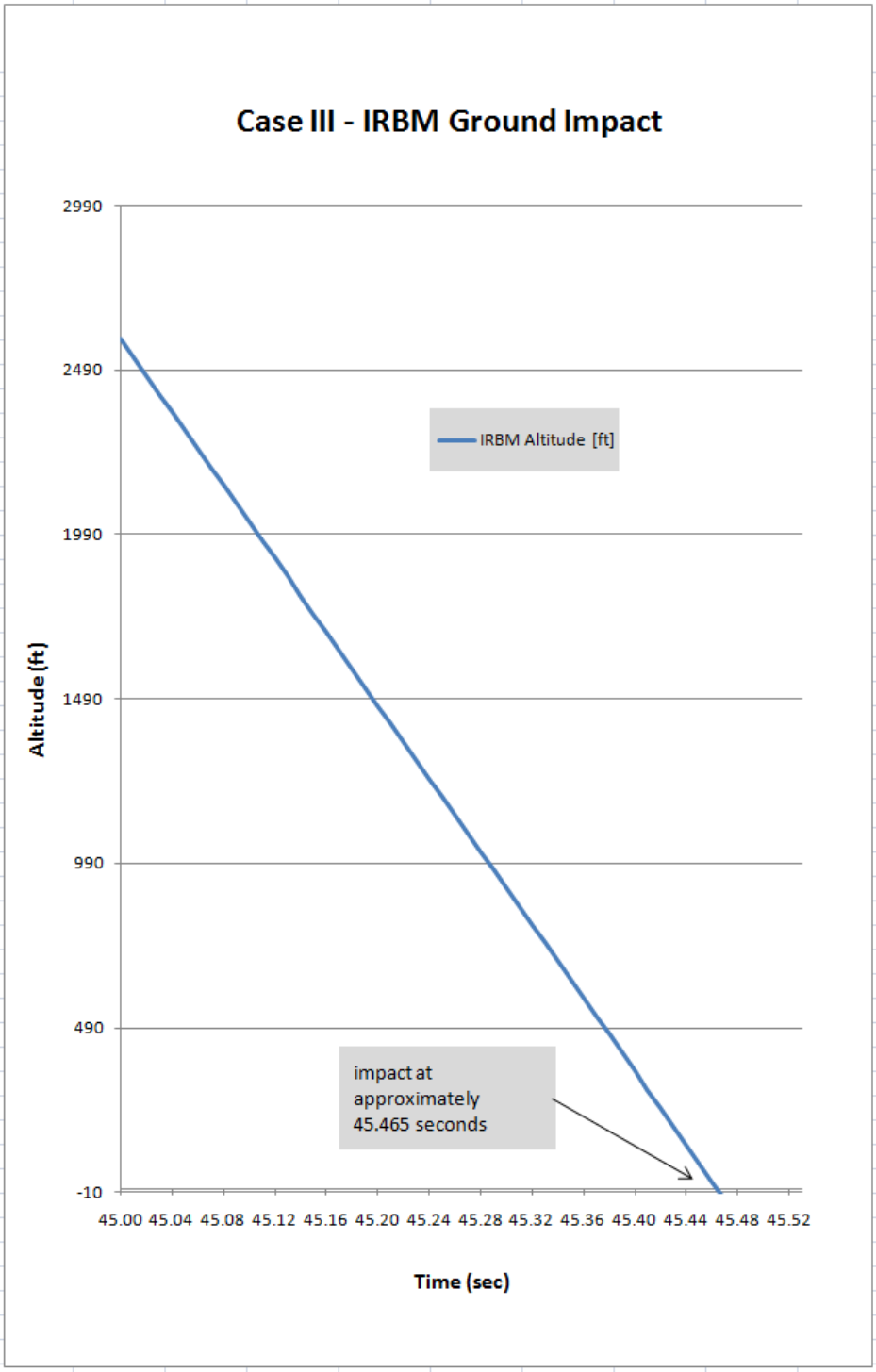
$2 \cdot mi = \frac{-g}{2} \cdot (t_{ILate} - d)^2 + v_{oPat} \cdot (t_{ILate} - d) + s_{oPat}$ solving for the required delay:

$$d := \begin{bmatrix} \frac{2 \cdot \left(\frac{g \cdot t_{ILate}}{2} - \frac{v_{oPat}}{2} + \frac{\sqrt{v_{oPat}^2 - 4 \cdot g \cdot mi + 2 \cdot g \cdot s_{oPat}}}{2} \right)}{g} \\ -\frac{2 \cdot \left(\frac{v_{oPat}}{2} - \frac{g \cdot t_{ILate}}{2} + \frac{\sqrt{v_{oPat}^2 - 4 \cdot g \cdot mi + 2 \cdot g \cdot s_{oPat}}}{2} \right)}{g} \end{bmatrix} = \begin{pmatrix} 42.25794 \\ -459.57654 \end{pmatrix} s$$

Appendix C – Case III

t start (sec) =	45		units
t inc (sec) =	0.01		sec
launch delay (sec) =		39.4	sec
initial height (ft) =	220000	0	ft
initial vel (ft/sec) =	-4107	8115	ft/sec

Time (IRBM) [sec]	IRBM Altitude [ft]
45.00	2583
45.01	2527
45.02	2471
45.03	2416
45.04	2360
45.05	2305
45.06	2249
45.07	2194
45.08	2138
45.09	2082
45.10	2027
45.11	1971
45.12	1916
45.13	1860
45.14	1804
45.15	1749
45.16	1693
45.17	1638
45.18	1582
45.19	1526
45.20	1471
45.21	1415
45.22	1359
45.23	1304
45.24	1248
45.25	1192
45.26	1137
45.27	1081
45.28	1026
45.29	970
45.30	914
45.31	859
45.32	803
45.33	747
45.34	692
45.35	636
45.36	580
45.37	525
45.38	469
45.39	413
45.40	358
45.41	302
45.42	246
45.43	190
45.44	135
45.45	79
45.46	23
45.47	-32
45.48	-88
45.49	-144
45.50	-200
45.51	-255
45.52	-311
45.53	-367



Case III Finding the root of the $h_{IRBM}(t)$ $0 \cdot mi = \frac{-g}{2} \cdot t^2 + v_{oIRBM} \cdot t + s_{oIRBM}$

$$t_{Impact} := \begin{bmatrix} \frac{2 \cdot \left(\frac{v_{oIRBM}}{2} + \frac{\sqrt{v_{oIRBM}^2 + 2 \cdot g \cdot s_{oIRBM}}}{2} \right)}{g} \\ -\frac{2 \cdot \left(\frac{\sqrt{v_{oIRBM}^2 + 2 \cdot g \cdot s_{oIRBM}}}{2} - \frac{v_{oIRBM}}{2} \right)}{g} \end{bmatrix} = \begin{pmatrix} 45.469 \\ -300.76793 \end{pmatrix} s$$

Appendix D - Clean Copy of Problem Statement

CCHS Math
Intermediate Algebra

Patriot Computer Lab – 50 Points
(Quadratic Application)

Name: _____
11/15/2010

Patriots Over Israel

On December 20, 2012, Iran launches an intermediate range ballistic missile (IRBM) carrying a WMD warhead at Israel. Israel is protected along it borders by US supplied Patriot missile batteries.

At t = 0 seconds, one of the US Patriot missile batteries detects the incoming IRBM warhead at a height of 220,000 ft and having a vertical downward velocity of 4107 ft / second. Assume the Patriot launches with an initial upwards velocity of 8115 ft/sec.

Case I: If a Patriot missile can be launched no earlier than 12 seconds after detection, determine the altitude and time (in seconds) after detection the Patriot can intercept the incoming IRBM warhead.

Case II: If the *minimum altitude* the IRBM can be intercepted is 2 miles, determine the *latest time* a Patriot can be launched to intercept an IRBM.

Case III: If a Patriot missile fails to intercept the IRBM, calculate the exact time of impact.

Include Excel graphs of both missiles' altitude vs. time in seconds showing where they intersect (the time and the height) for both the early and late launch cases.

Remember all ballistic motion for both the IRBM and the Patriot is governed by the equation:

$$h(t) = \left(\frac{-1}{2}\right) g t^2 + V_o t + S_o$$

Where:

h(t) is the height at any time t (seconds)
g = 32.2 ft/sec^2 [acceleration due to gravity]
Vo = is the initial velocity (ft/sec, downwards is -, + is upwards)
So is the initial height (ft) above the ground

Present you solution work (INCLUDING GRAPHS) in a neat, ORGANIZED, word-processed format with no handwriting. Include this sheet as the cover page. E-mail me a copy the Excel spreadsheets and completed document.

Explicitly identify the answers for Case I and Case II!

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